

THE MINIMAL VOLUME ORIENTABLE HYPERBOLIC 3-MANIFOLD WITH 4 CUSPS

KEN'ICHI YOSHIDA

Abstract

We prove that the 8_2^4 link complement is the minimal volume orientable hyperbolic manifold with 4 cusps. Its volume is twice of the volume V_8 of the ideal regular octahedron, i.e. $7.32\dots = 2V_8$. The proof relies on Agol's argument used to determine the minimal volume hyperbolic 3-manifolds with 2 cusps. We also need to estimate the volume of a hyperbolic 3-manifold with totally geodesic boundary which contains an essential surface with non-separating boundary.

1 Introduction

For hyperbolic 3-manifolds, their volumes are known as a topological invariant. The structure of the set of the volumes of hyperbolic 3-manifolds is known.

Theorem 1.1. (Jørgensen-Thurston) (Benedetti-Petronio [4, corollary E.7.1 and corollary E.7.5]) *The set of the volumes of orientable hyperbolic 3-manifolds is a well-ordered set of the type ω^ω with respect to the order of \mathbb{R} . The volume of an orientable hyperbolic 3-manifold with n -cusps corresponds to an n -fold limit ordinal.*

This theorem gives rise to the problem of determining the minimal volume orientable hyperbolic 3-manifolds with n cusps. The answers are known in the cases where $0 \leq n \leq 2$.

- In the case where $n = 0$ (closed manifold),
Gabai, Meyerhoff and Milley [9] showed that the Weeks manifold has the minimal volume. Its volume is $0.94\dots$
- In the case where $n = 1$,
Cao and Meyerhoff [6] showed that the figure-eight knot complement and the manifold obtained by the $(5,1)$ -Dehn surgery from the Whitehead link complement have the minimal volume. Their volume is $2.02\dots = 2V_3$, where V_3 is the volume of the ideal regular tetrahedron.
- In the case where $n = 2$,
Agol [2] showed that the Whitehead link complement and the $(-2,3,8)$ -pretzel link complement have the minimal volume. Their volume is $3.66\dots =$

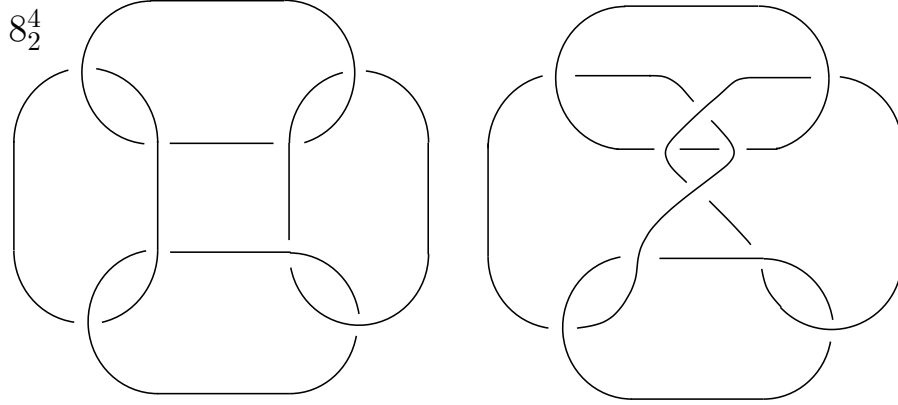


Figure 1: The 8_2^4 link and a link whose complement is homeomorphic to that of the 8_2^4 link

$$4 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = V_8, \text{ where } V_8 \text{ is the volume of the ideal regular octahedron.}$$

In the case where $n \geq 3$, Adams [1] showed that the volume of an n -cusped hyperbolic 3-manifold is not less than nV_3 . Agol [2] conjectured the following:

- In the case where $3 \leq n \leq 10$,
the minimally twisted hyperbolic chain link complement has the minimal volume.
- In the case where $n \geq 11$,
the $(n - 1)$ -fold covering of Whitehead link complement has the minimal volume.

In this paper, we prove this conjecture in the case where $n = 4$.

Theorem 1.2. *The minimal volume orientable hyperbolic 3-manifold with 4 cusps is homeomorphic to the 8_2^4 link complement. Its volume is $7.32\dots = 2V_8$.*

We remark that this link is not unique one to determine the complement. For example, the complement of the link on the right of Figure 1 is homeomorphic to the 8_2^4 link complement.

We will prove Theorem 1.2 in Sections 4 and 5. The proof owes much to Agol [2].

2 Review of Agol's argument

In this section, we set up some notation and review the argument used by Agol [2] to determine the minimal volume of 2-cusped hyperbolic 3-manifolds. We treat compact smooth 3-manifolds with boundary and corner. We only consider

surfaces in a compact 3-manifold which are properly embedded or contained in the boundary. Let $I = [0, 1]$.

Let M be a 3-manifold with boundary. For a properly embedded surface $X \subset M$, let $M \setminus\!\!\setminus X$ denote the path-metric closure of $M - X$. We will say that X is *essential* if X is incompressible and ∂ -incompressible and has no component parallel to the boundary. Essential surfaces are not assumed to be connected.

A finite volume orientable hyperbolic 3-manifold can be the interior of a compact 3-manifold with the boundary which consists of tori. Its boundary component is called a *cusps*. When we say a hyperbolic manifold in what follows, it often means this compact manifold. We also consider hyperbolic manifolds with totally geodesic boundary. In this case there may be *annular cusps* which adjoin the totally geodesic boundary. The *double* of a hyperbolic manifold M with totally geodesic boundary is the manifold obtained from two copies of M by gluing along the totally geodesic boundary. Then two annular cusps form one torus cusp in its double.

We introduce the notion of pared manifolds. It was defined by Thurston [18] to characterize a topological property of geometrically finite hyperbolic manifolds.

Definition 2.1. (Thurston [18, Section 7], Morgan [15, Definition 4.8]) A *pared manifold* is a pair (M, P) such that

- M is a compact orientable irreducible 3-manifold,
- $P \subset \partial M$ is a union of annuli and tori which are incompressible in M ,
- every abelian, noncyclic subgroup of $\pi_1(M)$ is peripheral with respect to P (i.e. conjugate to a subgroup of the fundamental group of a component of P), and
- every map $\phi : (S^1 \times I, S^1 \times \partial I) \rightarrow (M, P)$ which induces injective maps on the fundamental groups deforms, as maps of pairs, into P .

P is called the *parabolic locus* of the pared manifold (M, P) , and an annular component of P is called a *pared annulus*. We denote by $\partial_0 M$ the surface $\partial M - \text{int}(P)$.

A pared manifold (M, P) is called *acylindrical* if every map $\psi : (S^1 \times I, S^1 \times \partial I) \rightarrow (M, \partial_0 M)$ which induces injective maps on the fundamental groups deforms either into $\partial_0 M$ or into P .

Since a finite volume orientable hyperbolic 3-manifold is atoroidal, it is a pared manifold. Conversely the following holds:

Theorem 2.2. *Let (M, P) be an acylindrical Haken pared manifold, and assume that $\partial_0 M$ is incompressible. We assume that M is not a 3-ball, a $T^2 \times I$ or a solid torus. Then $M - P$ admits a finite volume hyperbolic structure with totally geodesic boundary $\partial_0 M$. This hyperbolic structure is unique up to isometry.*

Since the double DM of an acylindrical pared manifold (M, P) is atoroidal, DM admits a finite volume hyperbolic structure. Then the diffeomorphism

swapping the two copies of M can be taken to be an isometry. The fixed point set $\partial_0 M$ is totally geodesic [Leininger 13, Lemma 2.6].

When a hyperbolic manifold is cut along an essential surface, the obtained manifold is a pared manifold.

Lemma 2.3. (Agol [2, Lemma 3.2]) *Let M be a finite volume orientable hyperbolic 3-manifold, and ∂M be the parabolic locus P of M . Let $X \subset M$ be an essential surface. Then $(M \setminus X, P \setminus \partial X)$ is a pared manifold.*

Theorem 2.4. (JSJ decomposition for a pared manifold) (Jaco-Shalen [11], Johansson [12], Morgan [15, Section 11]) *Let (M, P) be a pared manifold such that $\partial_0 M$ is incompressible. There is a canonical set of essential annuli $(A, \partial A) \subset (M, \partial_0 M)$ called the characteristic annuli. It is characterized up to isotopy by the property that they are the maximal collection of non parallel essential annuli such that every other essential annulus $(B, \partial B) \subset (M, \partial_0 M)$ may be relatively isotoped to an annulus $(B', \partial B') \subset (M, \partial_0 M)$ so that $B' \cap A = \emptyset$. Then each complementary component $(L, \partial_0 L) \subset (M \setminus A, \partial_0 M \setminus \partial A)$ is one of the following types:*

1. $(T^2 \times I, (T^2 \times I) \cap \partial_0 M)$, where one of the boundary components $T^2 \times \partial I$ is a torus component of P .
2. $(S^1 \times D^2, (S^1 \times D^2) \cap \partial_0 M)$, which is a solid torus with annuli in the boundary.
3. $(I\text{-bundle}, \partial I\text{-subbundle})$, which is an I -bundle over a surface whose Euler characteristic is negative, and the I -bundle over the boundary is contained in $A \cup P$.
4. $(L, L \cap \partial_0 M)$, where L has no essential annuli whose boundary is contained in $L \cap \partial_0 M$.

A neighborhood of a torus component of P is either of type 1 or of type 4. One of the boundary components $T^2 \times \partial I$ of type 1 is a torus component of P , and the intersection of the other boundary component and $\partial_0 M$ is essential annuli in the torus. The intersection $(S^1 \times D^2) \cap \partial_0 M$ in a component of type 2 is essential annuli in $\partial(S^1 \times D^2)$. The union of components of type 3 is called *window*. A component of type 4 is acylindrical pared manifold $(L, L - \partial_0 M)$. The union of the components of type 4 is called *guts* and denoted by $\text{Guts}(M, P)$. A torus boundary component of the guts is a torus component of P .

The definition of guts in [2] is a bit different from ours. In [2] guts are defined to be the union of types 1, 2 and 4. The definition in [3] is same as ours, and it is appropriate for our purpose.

Let M be a finite volume orientable hyperbolic manifold. For an essential surface $X \subset M$, $(M \setminus X, P \setminus \partial X)$ is a pared manifold by Theorem 2.3. Therefore, we can define $\text{Guts}(X) = \text{Guts}(M \setminus X, P \setminus \partial X)$. Then the components of $\text{Guts}(X)$ admit hyperbolic structures with geodesic boundary by Theorem 2.2. Hence the volume $\text{vol}(\text{Guts}(X))$ is defined. This volume is not greater than the volume of M .

Theorem 2.5. (Agol-Storm-Thurston [3, Theorem 9.1]) *Let M be a finite volume orientable hyperbolic manifold, and $X \subset M$ be an essential surface. Then*

$$\text{vol}(M) \geq \text{vol}(\text{Guts}(X)) \geq \frac{V_8}{2} |\chi(\partial \text{Guts}(X))|.$$

Moreover, the refinement by Calegari, Freedman and Walker [5, Theorem 5.5] implies that M is obtained from ideal regular octahedra by gluing along the faces when the equality holds.

The estimate of $\text{vol}(\text{Guts}(X))$ from below in Theorem 2.5 follows from the following theorem.

Theorem 2.6. (Miyamoto [14, Theorem 5.2]) *Let M be a hyperbolic manifold with totally geodesic boundary. Then $\text{vol}(M) \geq \frac{V_8}{2} |\chi(\partial M)|$. Moreover, M is obtained from ideal regular octahedra by gluing along their faces when the equality holds.*

Lemma 2.7. *Let M be a finite volume orientable hyperbolic 3-manifold, and $X \subset M$ be a non-empty essential surface. Then each component of $\text{Guts}(X)$ has negative Euler characteristic.*

Proof. Since the Euler characteristic of every closed 3-manifold is 0, $\chi(\text{Guts}(X)) = \frac{1}{2} \chi(\partial \text{Guts}(X))$. Assume that there is a component L of $\text{Guts}(X)$ such that $\chi(L) \geq 0$. Since no component of $\partial \text{Guts}(X)$ is a sphere, $\chi(L) = 0$ and ∂L consists of tori. Since M is atoroidal, $\partial L \subset \partial M$. This implies $L = M$ by connectedness of M . This contradicts the fact that X is not empty. \square

This lemma implies that $\chi(\partial \text{Guts}(X)) \leq -4$ if $\text{Guts}(X)$ is not connected.

We will use annular compressions to obtain a surface whose guts is not empty.

Definition 2.8. Let $(X, \partial X) \subset (M, \partial M)$ be an essential surface in a 3-manifold. A *compressing annulus* is an embedding $i : (S^1 \times I, S^1 \times \{0\}, S^1 \times \{1\}) \hookrightarrow (M, X, \partial M)$ such that

- i_* induces injective maps on π_1 ,
- $i(S^1 \times I) \cap X = i(S^1 \times \{0\})$, and
- $i(S^1 \times \{0\})$ is not isotopic in X to ∂X .

An *annular compression* of $(X, \partial X) \subset (M, \partial M)$ is the surgery along a compressing annulus $i(S^1 \times I)$. Let U be a regular neighborhood of $i(S^1 \times I)$ in $M \setminus X$, and put $\partial_0 U = \partial U \cap (X \cup \partial M)$ and $\partial_1 U = \partial U - (X \cup \partial M)$. Then the surface $X' = (X - \partial_0 U) \cup \partial_1 U$ is called the annular compression of X . If X is essential, X' is also essential. We will say that $A_0 = \partial U \cap \partial M$ is the annulus in the boundary created by the annular compression (Figure 2). This annulus is not contained in the window of $M \setminus X'$.

Lemma 2.9. ([2, Lemma 3.3]) *Let M be a finite volume orientable hyperbolic manifold. Let $X \subset M$ be an essential surface. If X has a compressing annulus, let X' be the annular compression of X . Then the annulus in the boundary created by this annular compression is not contained in the window of $M \setminus X'$.*

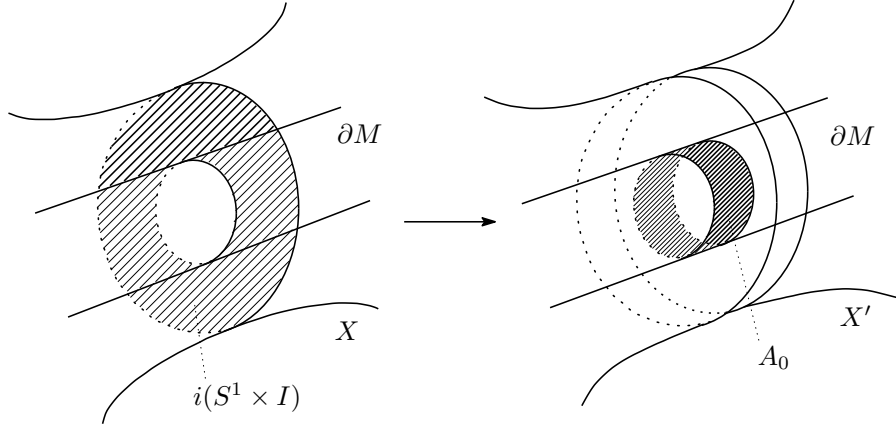


Figure 2: An annular compression

The following lemma is used in the proof of [2, Theorem 3.4]. Lemmas 2.9 and 2.10 imply that a torus or an annulus in the boundary is contained in the boundary of the gut regions after we perform annular compressions as many times as possible.

Lemma 2.10. *Let M and X be as above. We assume that a $T^2 \times I$ component or an $S^1 \times D^2$ component intersects a component T of ∂M . Then we can perform an annular compression for X toward T .*

The following theorem is a result in Culler-Shalen [8, Theorem 3]. We will use it to find an essential surface to start the proof of Theorem 4.2.

Theorem 2.11. *Let M be a finite volume orientable hyperbolic manifold with n cusps, and let $\partial M = T_1 \cup \cdots \cup T_n$, where T_i is a torus for $1 \leq i \leq n$. Let k be an integer such that $1 \leq k \leq n$. Then there is an essential surface $X \subset M$ such that $\partial X \cap T_i \neq \emptyset$ for $1 \leq i \leq k$ and $\partial X \cap (T_{k+1} \cup \cdots \cup T_n) = \emptyset$.*

3 Essential surfaces in 3-manifold with boundary

In this section we find an essential surface in a hyperbolic 3-manifold with geodesic boundary. Using this we will estimate the volume of a hyperbolic 3-manifold with geodesic boundary with at least 4 cusps. Essential surfaces are found by a homological argument for 3-manifolds, and it is not necessary to assume the hyperbolic structure.

Lemma 3.1. (Hatcher [10, Lemma 3.5]) *Let M be a compact orientable 3-manifold. Then the rank of the boundary homomorphism $\partial_* : H_2(M, \partial M; \mathbb{Q}) \rightarrow H_1(\partial M; \mathbb{Q})$ is half of the dimension of $H_1(\partial M; \mathbb{Q})$.*

Lemma 3.2. *Let L be an orientable hyperbolic 3-manifold with geodesic boundary S , with k annular cusps A_1, \dots, A_k and with $n - k$ torus cusps T_{k+1}, \dots, T_n ,*

where $1 \leq k \leq 3$ and $n \geq 4$. Assume that $\chi(S) = -2$. Then there is an essential surface $Y \subset L$ such that $Y \cap S = \emptyset$ and $[\partial Y] \neq 0 \in H_1(\partial L; \mathbb{Z})$.

Proof. The union $S' = S \cup A_1 \cup \dots \cup A_k$ is a closed surface of genus 2. We note that there are only two types of essential closed curves in S' , one separates S' and the other does not. There are no pairs of disjoint separating curves in S' .

We can take $k-1$ annuli of $\{A_1, \dots, A_k\}$ such that the complement of them is connected. The image of $\partial_* : H_2(L, \partial L; \mathbb{Q}) \rightarrow H_1(\partial L; \mathbb{Q})$ is an $(n-k+2)$ -dimensional subspace of $H_1(\partial L; \mathbb{Q}) \cong \mathbb{Q}^{2(n-k)+4}$ by Lemma 3.1. We consider the subspace V of $H_1(\partial L; \mathbb{Q})$ spanned by all the elements represented by curves in $A_1, \dots, A_{k-1}, T_{k+1}, \dots, T_n$. Since the dimension of V is $2(n-k) + (k-1)$, V intersects $\text{Im}(\partial_*)$ in a non-trivial subspace of $H_1(\partial L; \mathbb{Q})$. Hence there exists a non-zero element z in $H_2(L, \partial L; \mathbb{Q})$ such that $\partial_* z \neq 0$ and z belongs to V . By taking a multiple of z , there exists a non-zero element z' in $H_2(L, \partial L; \mathbb{Z})$ such that $\partial_* z' \neq 0$ and $\partial_* z'$ is represented by a union of closed curves in $A_1, \dots, A_{k-1}, T_{k+1}, \dots, T_n$. We can find an essential surface Y representing z' such that $\partial Y \subset A_1 \cup \dots \cup A_{k-1} \cup T_{k+1} \cup \dots \cup T_n$. \square

4 Estimate of volume

Now we are going to estimate the volume of a hyperbolic manifold with geodesic boundary. Lemma 3.2 and Theorem 4.1 imply that the volume of an orientable hyperbolic 3-manifold with 4 cusps and with geodesic boundary is not less than $2V_8$.

Theorem 4.1. *Let L be an orientable hyperbolic 3-manifold with geodesic boundary S . Suppose that there is an essential surface $Y \subset L$ such that $Y \cap S = \emptyset$ and $[\partial Y] \neq 0 \in H_1(\partial L; \mathbb{Z})$. Then there is an essential surface Y' such that $\chi(\partial \text{Guts}(L \setminus Y')) \leq -4$ and $\text{vol}(L) \geq 2V_8$.*

If $\chi(S) \leq -4$, then $\text{vol}(L) \geq 2V_8$ by Theorem 2.6. Hence we may assume that $\chi(S) = -2$. Let S' denote the surface which is the union of S and the annular cusps of L . ∂L consists of S' and the torus cusps of L .

We will find an essential surface Y' such that $\chi(\partial \text{Guts}(L \setminus Y')) \leq -4$. Then $\chi(\partial \text{Guts}(DL \setminus (DY' \cup S))) \leq -8$, where DL is the double of L (i.e. the hyperbolic manifold obtained from two copies of L by gluing along the geodesic boundary S) and DY' is the union of two copies of Y' in DL . Then Theorem 2.5 implies $\text{vol}(DL) \geq 4V_8$, and so $\text{vol}(L) \geq 2V_8$.

We will find a gut component intersecting S . For this we need to know how a window component intersects S .

Lemma 4.2. *Let L , S and Y be as above. Assume that S intersects a component $(J, \partial_0 J)$ of the window of $L \setminus Y$. Then $(J, \partial_0 J)$ is a product I -bundle and intersects S only on one component of the ∂I -bundle.*

Proof. Suppose that the base space of J is non-orientable. Since $\partial_0 J$ is connected, $\partial_0 J \subset S$. We take a simple closed curve α in J such that α is projected to an orientation-reversing loop in the base space of J . There is a simple closed curve β in $\partial_0 J$ such that $[\beta] = [\alpha]^2 \in \pi_1(DL) \subset \text{Isom}^+(\mathbb{H}^3)$. If β is homotopic

to the boundary of $\partial_0 J$, the base space of J is a Möbius band. It contradicts the definition of the window. Hence $[\beta] \in \pi_1(S) \subset \text{Isom}^+(\mathbb{H}^2)$ is hyperbolic element. The simple closed curve β is homotopic to a simple closed geodesic β' in S [16, Theorem 9.6.5]. But the fact that $[\beta'] = [\alpha]^2$ contradicts the fact that an element represented by a simple closed geodesic in a hyperbolic manifold has no roots [16, Theorem 9.6.2]. Therefore no twisted I -bundle intersects S .

Suppose that the base space of J is orientable and both components Q_0 and Q_1 of $\partial_0 J$ are contained in S . Since $\chi(Q_0) = \chi(Q_1) < 0$, there are (not necessarily simple) closed curves $\gamma_i \subset Q_i$ ($i = 0, 1$) such that γ_i is not homotopic to the boundary of Q_i and γ_0 and γ_1 are homotopic in L . Let γ'_i be the closed geodesic in Q_i homotopic to γ_i . Since L is totally geodesic, the two closed geodesics γ'_0 and γ'_1 are homotopic in L . It contradicts the uniqueness of the closed geodesic in a homotopy class. Therefore a product I -bundle intersects S on at most one side of the ∂I -bundle. \square

Proof of Theorem 4.1. Let Y_0 be an essential surface in L such that $Y_0 \cap S = \emptyset$ and $[\partial Y_0] \neq 0 \in H_1(\partial L; \mathbb{Z})$. Moreover let $|\chi(Y_0)|$ be minimal among the surfaces satisfying these conditions. Since L has no essential sphere, disk, torus or annulus, $\chi(Y_0) < 0$. Let $p : L \setminus Y_0 \rightarrow L$ be the natural projection.

(i) First we consider the case where S intersects a component $(J, \partial_0 J)$ of the window of $L \setminus Y_0$. Then $\chi(J)$ is equal to -1 or -2 . We will show that $\chi(J) = -1$. Assume that $\chi(J) = -2$. $S \cap p(J)$ is a 2-punctured torus or a 4-punctured sphere. (If it is a closed surface, $Y_0 \cap p(J)$ is a component of Y_0 which is parallel to S' . It contradicts that Y_0 is essential.) Let Y'_0 be the surface which is the union of $Y_0 - (Y_0 \cap p(J))$ and annuli (Figure 3). If there is an annulus in $L - Y_0$ whose boundary is two components of the frontier of $Y_0 - (Y_0 \cap p(J))$, we glue $Y_0 - (Y_0 \cap p(J))$ and this annulus (the upper of Figure 3). Since $Y_0 \cap p(J)$ is connected, the orientation matches. Otherwise, there is an annular cusp which is homotopic to the frontier of $Y_0 - (Y_0 \cap p(J))$. Then we can glue $Y_0 - (Y_0 \cap p(J))$ and the two annuli, where on of the boundary components of each annulus is contained in this annular cusp (the lower of Figure 3). Then $[Y'_0, \partial Y'_0] = [Y_0, \partial Y_0] \in H_2(L, \partial L; \mathbb{Z})$. We obtain an essential surface from Y'_0 by compressing if necessary. Then $|\chi(Y'_0)| < |\chi(Y_0)|$, contradicting the choice of Y_0 . Therefore $\chi(J) = -1$.

We will find an essential surface Y_1 such that S intersects only one component of the window of $L \setminus Y_1$. If S intersects only one component of the window of $L \setminus Y_0$ already, put $Y_1 = Y_0$. Suppose that S intersects two components $(J, \partial_0 J)$ and $(J', \partial_0 J')$ of the window of $L \setminus Y_0$. Let Y'_0 be the surface which is the union of $Y_0 - (Y_0 \cap p(J))$ and a surface in $p(J')$ (Figure 4). Then $[Y'_0, \partial Y'_0] = [Y_0, \partial Y_0] \in H_2(L, \partial L; \mathbb{Z})$. Note that since the orientation may not match, we cannot construct a surface as Figure 3. If Y'_0 is not essential, we obtain an essential surface smaller than Y_0 by compressing Y'_0 . Since it contradicts the choice of Y_0 , Y'_0 is essential.

Suppose that S intersects two components of the window of $L \setminus Y'_0$ again. Then one of these two components is contained in $p(J')$. We can perform the above construction again and remove a part of Y'_0 which is contained in the boundary of the window. Since no I -bundle can intersect S essentially along

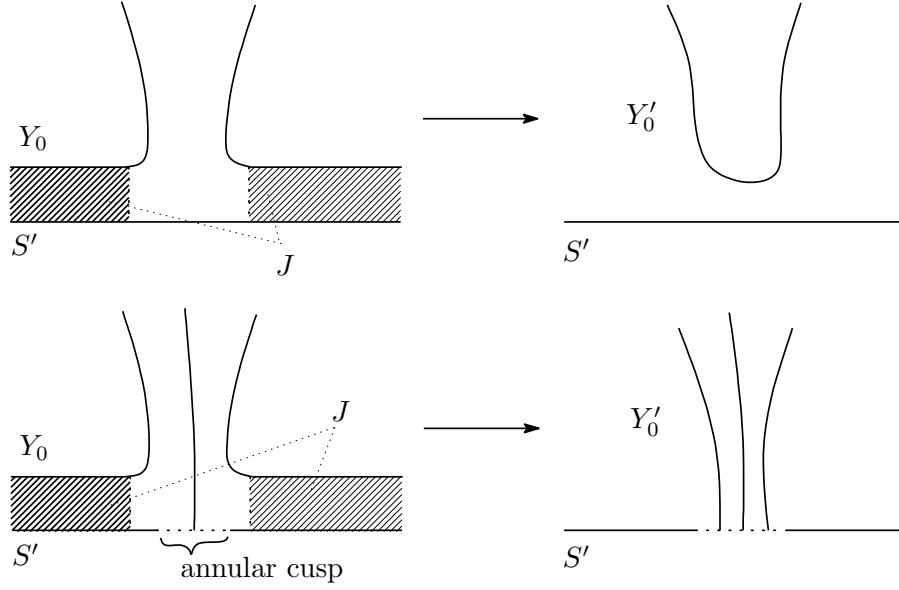


Figure 3: Constructions in the case where S intersects a component of the window whose Euler characteristic is -2

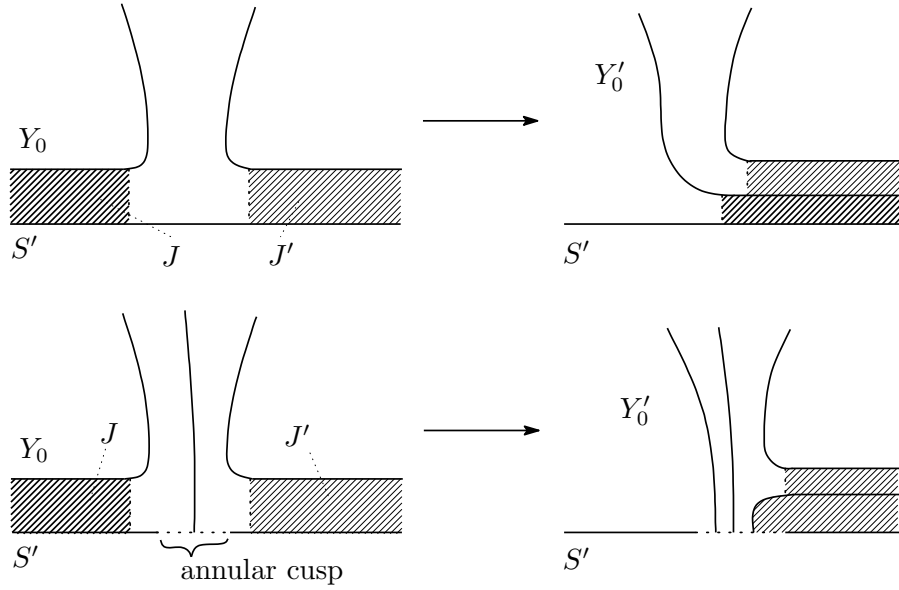


Figure 4: Constructions in the case where S intersects 2 components of the window whose Euler characteristics are -1

both components of the boundary by Lemma 4.2, the part of the obtained surface in $p(J')$ is not contained in the boundary of the component of the window which intersects S and lies on the same side as $p(J)$. Hence the above construction can be performed only finitely many times.

Let Y_1 be the essential surface obtained by performing the above construction as many times as possible. The Euler characteristic of the intersection of S and the window of $L \setminus Y_1$ equals -1 . Therefore the Euler characteristic of the intersection of S and $\text{Guts}(L \setminus Y_1)$ is equal to -1 . In particular, $\text{Guts}(L \setminus Y_1) \neq \emptyset$.

We will find an essential surface Y_2 such that $\chi(\partial \text{Guts}(L \setminus Y_2)) \leq -4$. If $\chi(\partial \text{Guts}(L \setminus Y_1)) \leq -4$, put $Y_2 = Y_1$. Suppose that $\chi(\partial \text{Guts}(L \setminus Y_1)) = -2$. Since the Euler characteristic of $\partial \text{Guts}(L \setminus Y_1) - S'$ is equal to -1 , it is either a 1-punctured torus or a 3-punctured sphere.

Suppose that $\partial \text{Guts}(L \setminus Y_1) - S'$ is a 1-punctured torus. Then $\partial \text{Guts}(L \setminus Y_1) - S'$ can contain a pared annulus, and $Y_1 \cap \partial \text{Guts}(L \setminus Y_1)$ is a 1-punctured torus or a 3-punctured sphere. If $Y_1 \cap \partial \text{Guts}(L \setminus Y_1)$ is a 1-punctured torus, let Y'_1 be the surface which is the union of $Y_1 - (Y_1 \cap \partial \text{Guts}(L \setminus Y_1))$ and a surface in $p(J')$ (Figure 5). If $Y_1 \cap \partial \text{Guts}(L \setminus Y_1)$ is a 3-punctured sphere, we obtain the surface \tilde{Y}_1 by modifying Y_1 around the pared annulus in $\partial \text{Guts}(L \setminus Y_1) - S'$ (Figure 6). Here $\tilde{Y}_1 \cap \partial \text{Guts}(L \setminus Y_1)$ is a 1-punctured sphere. Thus we obtain an essential surface Y'_1 as the union of $\tilde{Y}_1 - (\tilde{Y}_1 \cap \partial \text{Guts}(L \setminus Y_1))$ and a surface in $p(J')$ (Figure 5).

Suppose that $\partial \text{Guts}(L \setminus Y_1) - S'$ is a 3-punctured sphere. $\partial \text{Guts}(L \setminus Y_1) - S'$ does not contain a pared annulus. Let Y'_1 be the surface which is the union of $Y_1 - (Y_1 \cap \partial \text{Guts}(L \setminus Y_1))$ and a surface in $p(J')$.

We have obtained a surface $\partial Y'_1$ in these ways. Then $[Y'_1, \partial Y'_1] \neq [Y_1, \partial Y_1] \in H_2(L, \partial L; \mathbb{Z})$ in general, but $[\partial Y'_1] = [\partial Y_1] \neq 0 \in H_1(\partial L; \mathbb{Z})$. Since $|\chi(Y_1)| = |\chi(Y_0)|$, Y_1 is essential.

Since $Y_1 \cap \partial \text{Guts}(L \setminus Y_1)$ cannot be contained in the window of $L \setminus Y'_1$, $S \cap \partial \text{Guts}(L \setminus Y_1)$ is not contained in the window of $L \setminus Y'_1$. Hence we can consider that $\text{Guts}(L \setminus Y'_1)$ contains $S \cap \partial \text{Guts}(L \setminus Y_1)$. Since Y_1 is essential, the added surface in the window is not contained in $Y'_1 \cap \partial \text{Guts}(L \setminus Y'_1)$. Hence the above construction can be performed only finitely many times.

Let Y_2 be the essential surface obtained by performing the above construction as many times as possible. Then $\chi(\partial \text{Guts}(L \setminus Y_2))$ is no longer equal to -2 , and $\chi(\partial \text{Guts}(L \setminus Y_2)) \leq -4$.

(ii) Suppose that S intersects no component of the window of $L \setminus Y_0$. Then $\chi(\text{Guts}(L \setminus Y_0) \cap S) = -2$. Assume that $\chi(\partial \text{Guts}(L \setminus Y_0)) = -2$. $\partial \text{Guts}(L \setminus Y_0)$ is a closed surface which is the union of a surface in S and annuli. since L is atoroidal, $\partial \text{Guts}(L \setminus Y_0)$ contains the closed surface S' . Hence $\partial \text{Guts}(L \setminus Y_0)$ consists of S' and some torus cusps of L . The connectivity of L implies that $L = \text{Guts}(L \setminus Y_0)$. It contradicts that Y_0 is non-empty. Therefore $\chi(\partial \text{Guts}(L \setminus Y_0)) \leq -4$. \square

We prove the essential part of Theorem 1.2.

Theorem 4.3. *Let M be an orientable hyperbolic manifold with 4 cusps. Then $\text{vol}(M) \geq 2V_8$. Moreover, if $\text{vol}(M) = 2V_8$, M is obtained from two ideal regular*

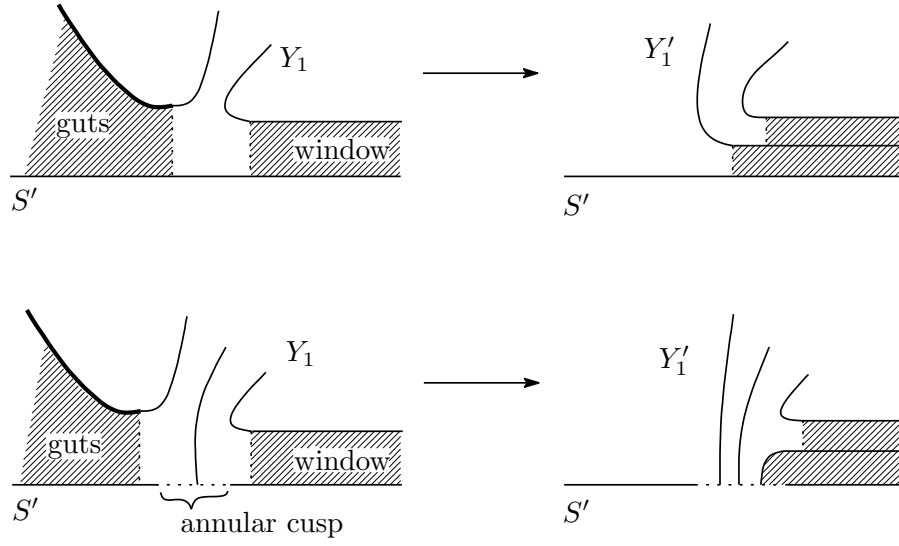


Figure 5: Constructions in the case where $\partial\text{Guts}(L \setminus Y_1) = -2$

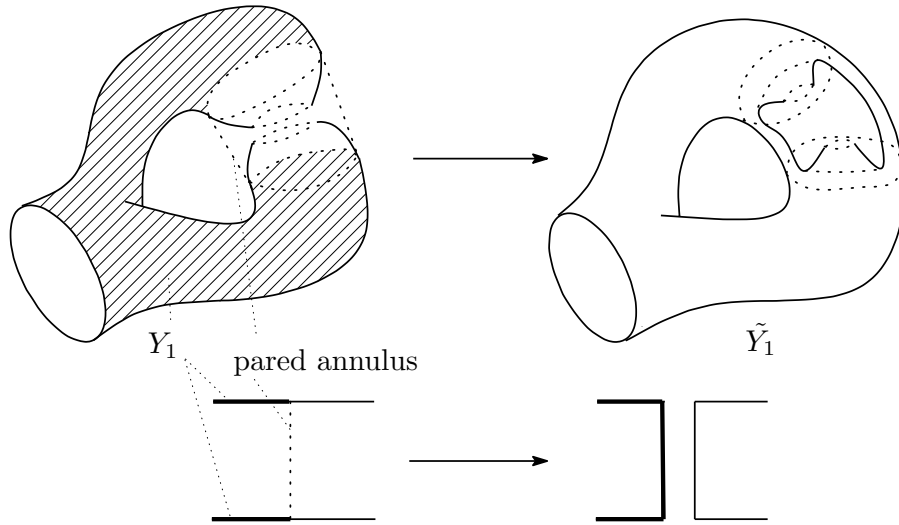


Figure 6: A construction around a pared annulus in $\partial\text{Guts}(L \setminus Y_1) - S'$

octahedra by gluing along the faces.

Proof. It is sufficient to find an essential surface $X \subset M$ such that $\chi(\partial \text{Guts}(X)) \leq -4$. Then Theorem 4.3 follows from Theorem 2.5.

Let T_1, \dots, T_4 be the cusps of M . We take an essential surface X_0 such that $X_0 \cap T_1 \neq \emptyset$ and $X_0 \cap T_i = \emptyset$ ($2 \leq i \leq 4$) by Theorem 2.11. We perform annular compressions for X_0 as many times as possible to obtain an essential surface X_1 . When annular compression is performed, the number of boundary components of the surface increases and its Euler characteristic does not change. Since the Euler characteristic of each component of the obtained essential surface is negative, annular compressions can be performed only finitely many times.

We will show that $\text{Guts}(X_1)$ intersects T_2, \dots, T_4 . Let k be the number of the cusps intersecting X_1 ($1 \leq k \leq 4$). Let T_1, \dots, T_k be the cusps intersecting X_1 . Let A_2, \dots, A_k be the annuli in $T_2 \setminus \partial X_1, \dots, T_k \setminus \partial X_1$ created by the last annular compressions to T_2, \dots, T_k . Since there are no compressing annuli any more, Lemma 2.10 implies that A_2, \dots, A_k are not contained in a solid torus component of the JSJ decomposition of $M \setminus X_1$ and T_{k+1}, \dots, T_4 are not contained in a $T^2 \times I$ component of it. Since compressing annuli to different cusps can be taken disjointly, we may change the order of annular compressions to different cusps. By Lemma 2.9, A_2, \dots, A_k are not contained in the window of $M \setminus X_1$. Therefore $A_2, \dots, A_k, T_{k+1}, \dots, T_4 \subset \partial \text{Guts}(X_1)$.

If $\text{Guts}(X_1)$ is not connected, then $\chi(\partial \text{Guts}(X_1)) \leq -4$ as desired. Suppose that $\text{Guts}(X_1)$ is connected. Then $A_2, \dots, A_k, T_{k+1}, \dots, T_4$ are contained in one component N of $M \setminus X_1$. We will find an essential surface X_2 such that $\partial \text{Guts}(X_2)$ contains at least 4 pared components.

(i) Suppose that $(T_1 \setminus \partial X_1) \cap N \neq \emptyset$. If $N = \text{Guts}(X_1)$, let A_1 be an annulus which is a component of $(T_1 \setminus \partial X_1) \cap N$. Otherwise let A_1 be an separating annulus of the JSJ decomposition intersecting $\text{Guts}(X_1)$. In either case, A_1 is a pared annulus of $\text{Guts}(X_1)$ different from A_2, \dots, A_k . Then it is sufficient to put $X_2 = X_1$.

(ii) Suppose that $(T_1 \setminus \partial X_1) \cap N = \emptyset$. Let $X'_1 = X_1 \cap p(N)$, where $p : M \setminus X_1 \rightarrow M$ is the natural projection. Then X'_1 is an essential surface in M and $T_1 \cap X'_1 = \emptyset$. X'_1 is the union of the components of X_1 intersecting N . If we cannot perform an annular compression for X'_1 to T_1 , $\text{Guts}(X'_1)$ contains a neighbourhood of T_1 which is in the complement of N . Since $\text{Guts}(X'_1)$ is not connected, $\chi(\partial \text{Guts}(X'_1)) \leq -4$. Then it is sufficient to put $X_2 = X'_1$.

If we can perform an annular compression for X'_1 to T_1 , we obtain X_2 by performing annular compressions to T_1 as many times as possible. Let A_1 be the innermost annulus in T_1 . Since X_1 is obtained by performing annular compressions as many times as possible, there is no compressing annulus for X'_1 to $A_2, \dots, A_k, T_{k+1}, \dots, T_4$ in $p(N)$. Hence there is no compressing annulus for X_2 to $A_2, \dots, A_k, T_{k+1}, \dots, T_4$ in $p(N)$. Since the surface which is obtained by filling X'_1 with A_2, \dots, A_k consists of components of a surface in the process of the annular compression from X_0 to X_1 , it is essential. This implies that A_1, \dots, A_k is not contained in the window of $M \setminus X_2$ by Lemma 4.3. Therefore $A_1, \dots, A_k, T_{k+1}, \dots, T_4 \subset \text{Guts}(X_2)$.

Finally, we will find an essential surface X such that $\chi(\partial \text{Guts}(X)) \leq -4$. If

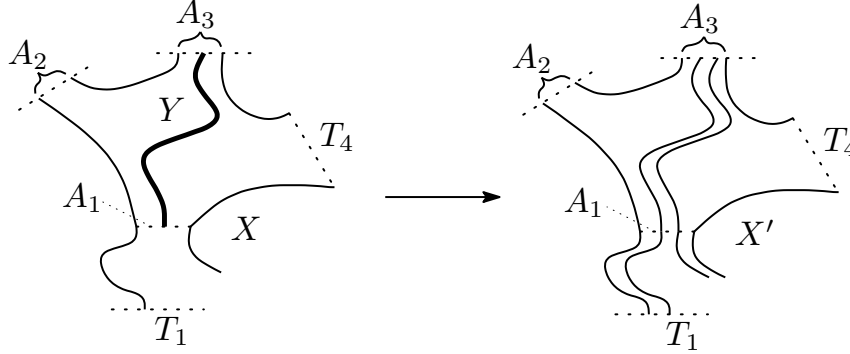


Figure 7: A construction of an essential surface the boundary of whose guts is no more than -4

$k = 4$, the 4 annuli A_1, \dots, A_4 are disjoint and not homotopic each other in the non-torus components of $\partial\text{Guts}(X_2)$. This implies that $\chi(\partial\text{Guts}(X_2)) \leq -4$. Then It is sufficient to put $X = X_2$.

If $1 \leq k \leq 3$, $\text{vol}(\text{Guts}(X_2)) \geq 2V_8$ by Theorem 4.1. Therefore $\text{vol}(M) \geq 2V_8$ by Theorem 2.5. But we need to find X in order to prove that M is obtained from 2 octahedra when the equality holds. Lemma 3.2 and Theorem 4.1 imply that there is an essential surface Y in $\text{Guts}(X_2)$ such that $\chi(\partial\text{Guts}(\text{Guts}(X_2) \setminus Y)) \leq -4$. Then Y intersects some of $A_1, \dots, A_k, T_{k+1}, \dots, T_4$, where $A_2, \dots, A_k, T_{k+1}, \dots, T_4$ is contained in ∂M . If A_1 is contained in ∂M or does not intersect Y , $X_3 \cup Y$ is properly embedded in M . Since $\text{Guts}(X_2 \cup Y) = \text{Guts}(\text{Guts}(X_2) \setminus Y)$, $\chi(\partial\text{Guts}(X_2 \cup Y)) \leq -4$. Then it is sufficient to let $X = X_2 \cup Y$. If A_1 is contained in the interior of M and intersects Y , $X_3 \cup Y$ is not properly embedded in M . Suppose that $A_1 \cap Y$ is the union of l simple closed curves. Let X be the union of 2 surfaces parallel to Y , $X_2 \cap \partial\text{Guts}(X_2)$ and $l + 1$ times of $X_2 - \partial\text{Guts}(X_2)$ (Figure 7). Since $\text{Guts}(X)$ is homeomorphic to $\text{Guts}(\text{Guts}(X_2) \setminus Y)$, $\chi(\partial\text{Guts}(X)) \leq -4$. \square

5 Realization of hyperbolic manifold

In this section we will prove that an orientable hyperbolic 3-manifold obtained from 2 ideal regular octahedra by gluing along the faces is homeomorphic to the complement of the 8_2^4 link. This completes the proof of Theorem 1.2.

Thurston calculated the volume of the complement of the 8_2^4 link in [17, Ch. 6, Example 6.8.6] and it is equal to $2V_8$. Moreover, SnapPy [7] has the list of the orientable hyperbolic 3-manifolds obtained from 8 ideal regular tetrahedra by gluing along the faces. These imply the uniqueness of the minimal volume orientable hyperbolic 3-manifold with 4 cusps, but we prove it here by an elementary argument examining the possible ways of gluing along the faces of 2 octahedra.

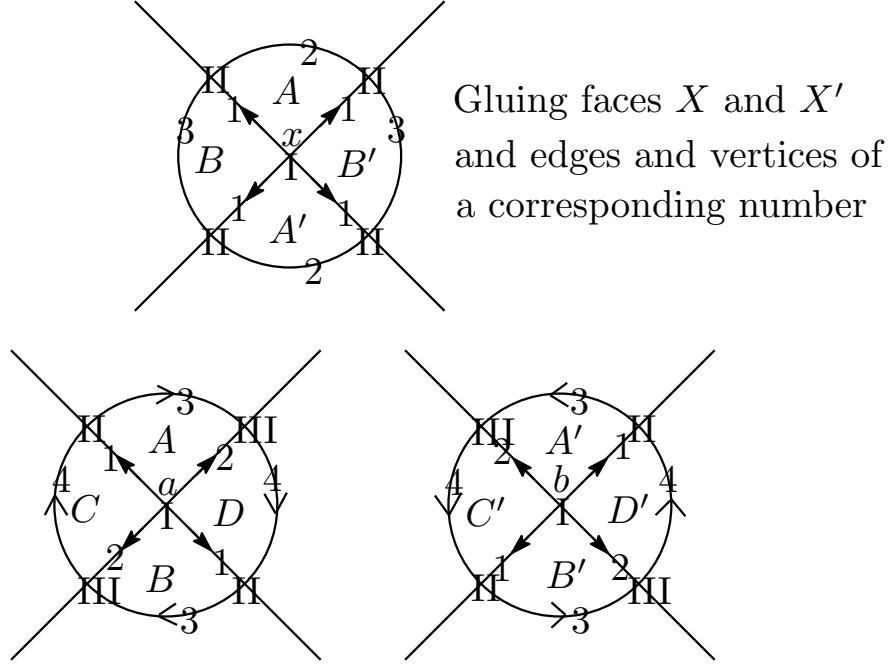


Figure 8: Gluing of the faces around vertices which are glued together

The 12 vertices of the 2 octahedra correspond the 4 cusps of the hyperbolic manifold. We look at the number of vertices corresponding to each cusp. Since the edge angles of an ideal regular octahedron are right angles, 4 edges of the 2 octahedra should be glued together.

Claim 5.1. *If there is a cusp consisting of 1 vertex x , the faces around x are glued as in the upper part of Figure 8. If there is a cusp consisting of 2 vertices a and b , the faces around a and b are glued as in the lower part of Figure 8.*

Proof. If there is a cusp consisting of 1 vertex x , the 4 edges around x are glued together, and each face around x is glued with the opposite face.

Suppose there is a cusp consisting of 2 vertices a and b . Assume that a and b are contained in one octahedron. If a and b are adjacent, no edges can be glued with the edge between a and b . If b is opposite to a , we can glue no pairs of faces which are contained in different octahedra. This contradicts the connectivity. Hence a and b are contained in different octahedra.

We consider how the 8 edges around a and b are glued. Since a and b are glued, the 4 edges around a cannot be glued together. If 3 edges around a and 1 edge around b are glued together, 2 adjacent faces around a are glued (the left of Figure 9). Then the edge between the 2 faces can be glued with no edges. Hence 2 edges around a and 2 edges around b are glued together. Assume that adjacent edges around a are glued. Let x and y be the vertices opposite to a and b respectively. If x and y form 2 cusps with themselves, there are 2 edges

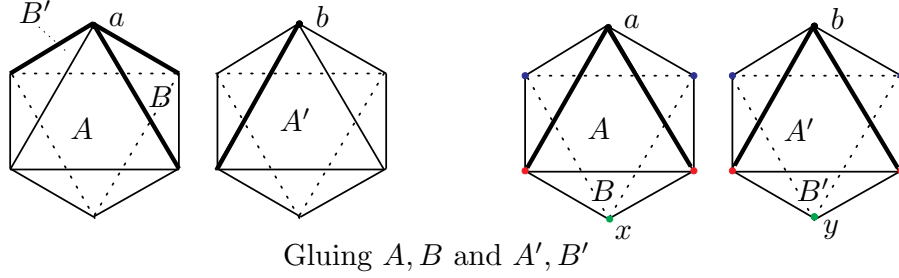


Figure 9: An impossible example in the case where 2 vertices form a cusp

glued with no other edges. Since there are 4 cusps, there is a cusp consisting of x and y . There are 2 edges glued with no other edges even in this case (the right of Figure 9). Therefore opposite edges around a are glued and the way of gluing is determined. \square

Claim 5.2. *There is no cusp consisting of 3 vertices.*

Proof. Assume that there is a cusp consisting of 3 vertices a, b and c . If a, b and c are vertices of one octahedron, 2 positions are possible (the left of Figure 10). If a, b and c are the vertices of one face, this face cannot be glued with another face. Otherwise, at least one of a, b and c is contained in a face of the octahedron containing a, b and c . This implies that no pair of faces of different octahedra can be glued. Hence a, b and c are not contained in one octahedron. We assume that b and c are contained in one octahedron without loss of generality. Then 2 positions are possible (the right of Figure 10). If b and c are adjacent, no edges can be glued with the edge between b and c . Hence c are opposite to b . Let x be the vertex opposite to a . Since only the 4 faces around x do not contain a, b or c , the 4 faces cannot be glued with any faces of the other octahedron.

Assume that x does not form a cusp with itself. Suppose that adjacent faces around x are glued. Then the 5 vertices except a of the octahedron containing a are glued together. There are 2 vertices y and z which form 2 cusps with themselves on the octahedron containing b and c . The 4 vertices around y are glued together by Claim 5.1 (Figure 11). This contradicts that b is glued only with a and c . Hence opposite faces around x are glued.

Suppose that opposite faces A and B around x are glued twistedly, i.e. the 2 vertices corresponding with x in A and B are not glued. Then 2 opposite vertices on the octahedron containing a are glued with x . Since the 5 vertices except a of the octahedron containing a cannot be glued together, the other faces C and D around x are glued twistedly. The 4 faces around a are glued with faces of the other octahedron because of the correspondence of the vertices and the fact that adjacent faces around a cannot be glued. Hence there is a vertex which forms a cusp with itself on the octahedron containing b and c (Figure 12). This contradicts that b is glued only with a and c .

Hence x forms a cusp with itself. Since 4 edges are glued together, the faces

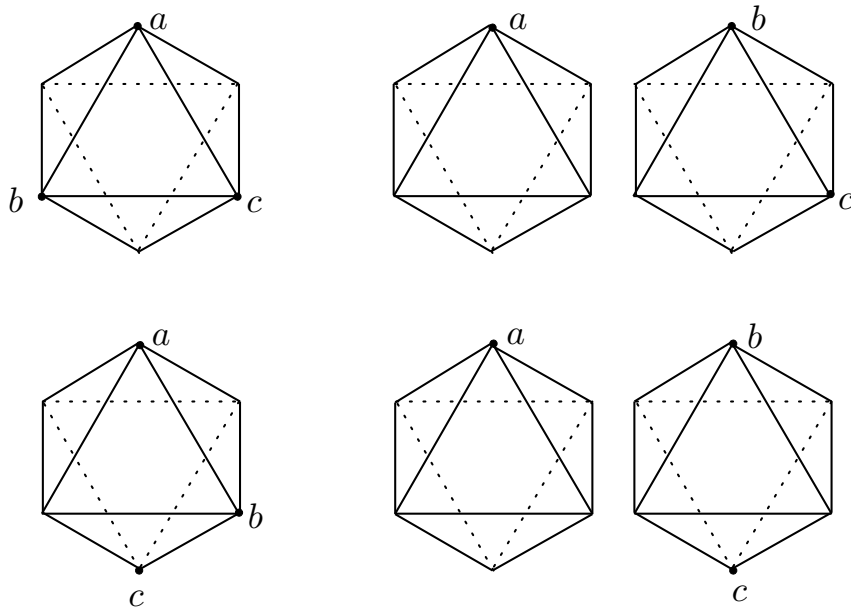


Figure 10: Positions of 3 vertices

These points form 2 cusps with themselves.

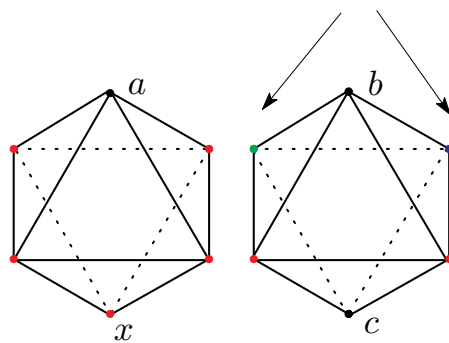


Figure 11: It is impossible that 5 vertices of an octahedron are glued together.

This point forms a cusp with itself.

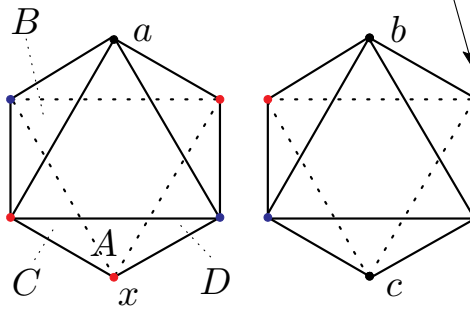


Figure 12: It is impossible that opposite faces are glued twistedly.

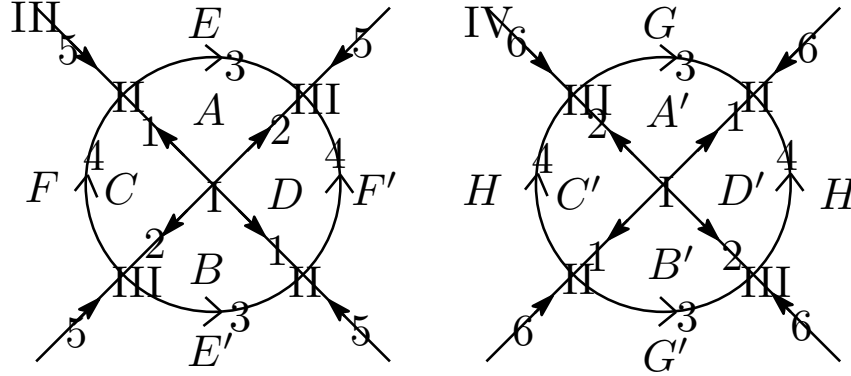


Figure 13: The way of gluing of the octahedra (i)

around a are glued with faces of the other octahedron. At least 3 vertices of the octahedron containing b and c are glued with the 4 vertices except a and x . Since we must obtain 4 cusps, there is a vertex which form a cusp with itself. It is contradiction. \square

Claim 5.2 implies that there is a cusp consisting of 1 or 2 vertices. Suppose that there is a cusp consisting of 1 vertex x . The 4 vertices around x are glued together. The 4 faces A, B, C and D around the vertex a opposite to x are glued with faces of the other octahedron. a is glued with only 1 vertex b because of Claim 5.2 and the fact that 7 vertices are glued. Since the 8 vertices around a and b are glued together, the vertex y opposite to b forms a cusp with itself. The numbers of the vertices corresponding the cusps are 1, 1, 2 and 8. By Claim 5.1 the way of gluing is determined as Figure 13 (i).

Suppose that there is no cusp consisting of 1 vertex. Then there is a cusp consisting of 2 vertices a and b . A, A', B, B', C, C', D and D' around a and b

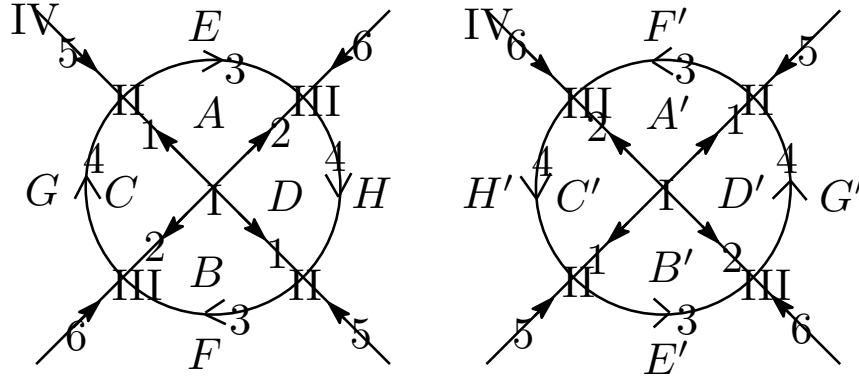


Figure 14: The way of gluing of the octahedra (ii)

are glued as Figure 8. Since no cusp consists of 1 vertex, the 2 vertices x and y opposite to a and b respectively are glued together. The numbers of the vertices corresponding the cusps are 2, 2, 4 and 4. The face E adjacent to A is glued with the face E' adjacent to B' because of the correspondence of the vertices and edges. The way of gluing is determined as Figure 14 (ii).

Both cases of (i) and (ii) give homeomorphic spaces by Figure 15 and they are the 8_2^4 link complements by Figure 16.

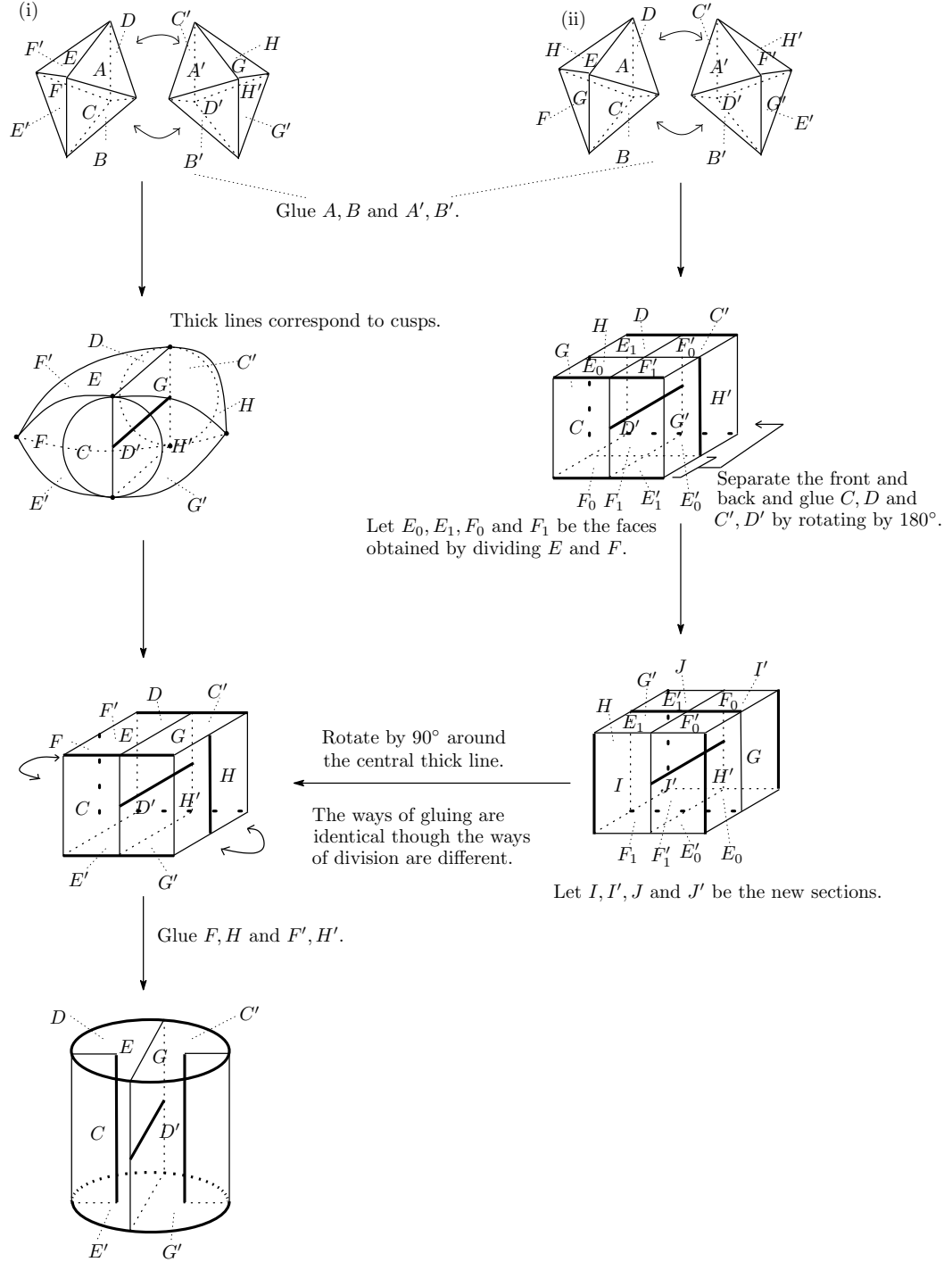


Figure 15: Gluing of the octahedra 1

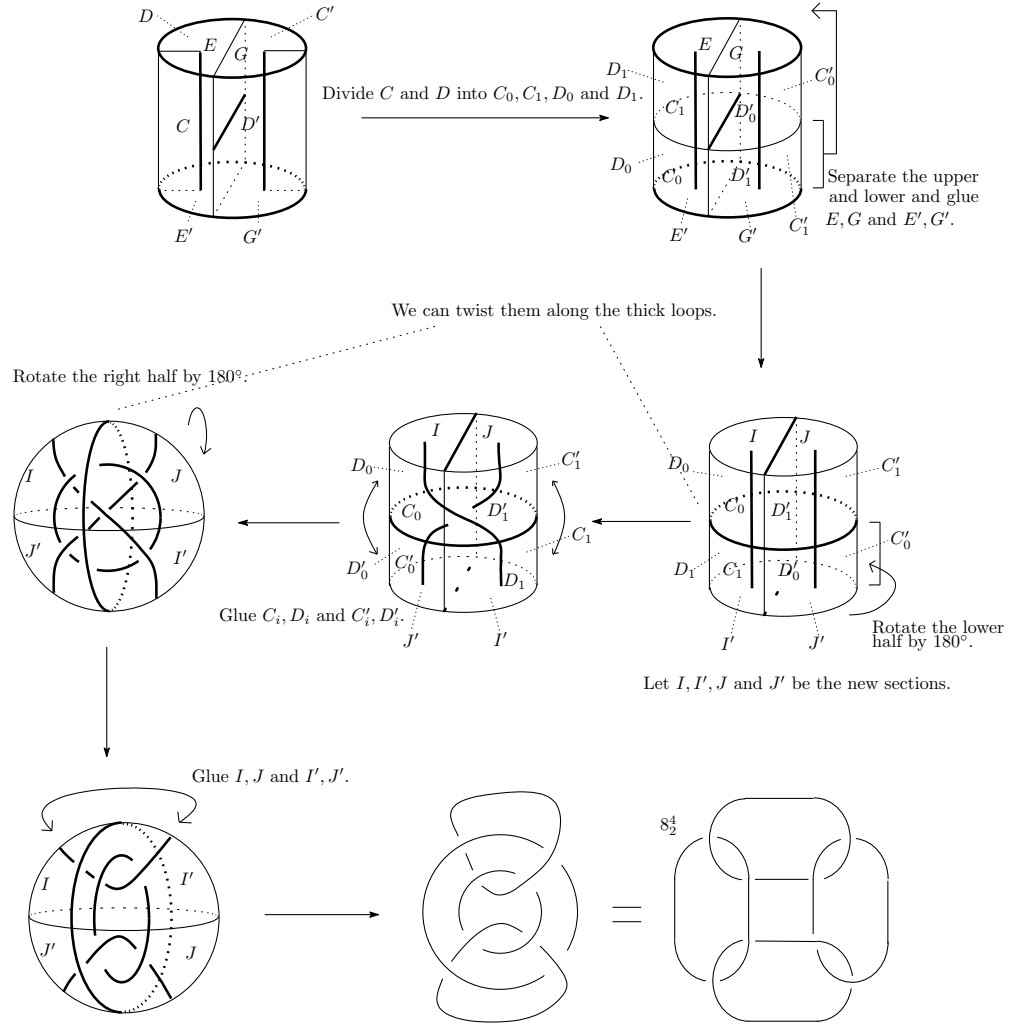


Figure 16: Gluing of the octahedra 2

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GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO,
3-8-1 KOMABA, MEGURO-KU, TOKYO 153-8914, JAPAN.

E-mail address: kyoshida@ms.u-tokyo.ac.jp